TIME-DOMAIN NUMERICAL BUFFETING RESPONSE PREDICTION

I. THEORY

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1. INTRODUCTION

(1) Time domain buffeting response analysis is considering as the state-of-the-art approach in only recently several years, because of its advantages: i) treatment with both aerodynamics nonlinearity and geometrical nonlinearity that is common sense for wind-induced vibrations of very long-span bridges and ii) close relation to mechanical controls of wind-induced vibrations. Main idea for time domain buffeting response analysis expresses in following figure:

![Diagram showing the process of time domain buffeting response analysis]

(2) Literature review on time domain buffeting response analysis

- Wagner (1925) introduced the time-domain indicial function that expresses relationship of unsteady aerodynamic lift forces on angle of attack and force coefficient. Wagner’s indicial function can be expressed under form of polynomial one on the Laplace variable.
- Kusner (1936) developed the Wagner’s problem to solve unsteady aerodynamic response of airfoil under uniform gust of airfoil, whereas Sears (1941) developed solution for vertical gust.
o Liepmann (1952) developed theoretical buffeting analysis for subject of airplane’s wing that based on both spectrum analysis and statistical computation

o Davenport (1962) firstly introduced a framework method for buffeting analysis of bridge and tower structures, that combined by spectrum analysis and statistical computation in the modal space. Especially, he invented so-called joint acceptance function that . However, single mode-based response prediction has been taken into consideration, cross coupling between modes neglected.

o Coupled buffeting and flutter analysis: Reasons for simultaneously coupled buffeting and flutter analysis are that i) Turbulence or unsteady wind is nature of wind, ii) In high wind velocity, flutter and buffeting potentially occur simultaneously and iii) Aerodynamic forces by quasi-steady theory (= Time invariant forces [Static] + Time variant forces [Buffeting] + Self excited forces [Flutter])

o Matsumoto et al.(1994): Influence of unsteady self-excited forces has been taken into consideration of buffeting response prediction.
o Time domain buffeting response analysis through the indicial function: Wagner(1925), Kussner(1936), Scanlan (1976), Scanlan (1993), Borri(2005)


o Convolution integrals has been firstly applied for wind-induced instability analysis by TY.Lin et al.(1988) in which time-domain function can be expressed under sum of response impulse function (RIF). This convolution integral can cope with so-called ‘memory effect of fluid’.

(3) Procedure of time domain buffeting analysis consists of: 1) Time-series simulation of natural wind and 2) time-dependant transformation of unsteady aerodynamic force (flutter and buffeting) by using indicial function (more concretely, rational function approximation), that combined with 3) time-history computer-aided structural programs (geometrical nonlinear behavior can be included in analysis) are main points in recently time-domain aerodynamic analysis of bridges

Thus three interrested questions:

1) How to simulated time series wind field?

2) How to model time domain unsteady aerodynamic forces (Linear aerodynamic forces and Non-linear aerodynamic forces)

3) How to solve the time-history analysis and prediction for full-scale bridge analysis
2. TIME SERIES WIND FLUCTUATION SIMULATION

2.1. Methodological background

(1) It is assumed that wind turbulence (or fluctuating velocity components) may be described as the wide-band stationary Gaussian (Normal-distributed) random processes with zero-mean value and positive one-sided power spectral density (PSD) functions.

(2) Time-series turbulence processes must be simulated at external-loading-imposed nodes on bridge based on target processes.

(3) Two types of target processes commonly are used in simulation techniques:
   o The simulated time-series processes indirectly obtained (Indirect approach) from the target power spectral density functions, known as spectral representation
   o The simulated time-series processes directly obtained from the target time-series processes, known as time-series representation. This technique is high efficient and time-saving, but high sense in system stability must be paid much attention

(4) Literature review on turbulence simulation technique by spectral representation
Shinozuka (1971) firstly introduced the simulation techniques for spectral representation of stationary Gaussian random processes, such as Cholesky decomposition and Modal factorization.


Yang et al. (1997); N.N.Minh et al. (1999); Cao et al. (2000): Computational procedure and numerical applications for long-span bridges.

(5) Literature review on turbulence simulation technique by time series representation

The time-series representation for turbulence wind simulation can be expressed by i) auto-regressive technique (AR), ii) moving-average technique (MA) and combined auto-regressive and moving average (ARMA) technique.

Auto-regressive and moving-average (ARMA) method has presented for time series wind simulation by [Samaras et al. (1985), Migolet et al. (1987), Li et al. (1990), Kareem et al. (1992), Meada et al. (1992)].
(6) The spectral representation methods are usually associated with either the Cholesky’s decomposition or Modal factorization from given spectral density matrix to generate sample random variables. These methods can create unconditionally stable results, however, simulation procedure is very time-consumed computation due to decomposition of spectral density matrix, numerous simulated velocities at various points on bridge and presence of sample random variables in computation. The time series representation (ARMA) offer more effective simulation procedure with time-saving, however, they require more attention on stability of system and computational algorithm.

Assumptions on wind field and turbulence simulation: i) Wind field is unidirectional, stationary and homogenous and ii) Wind field is Gaussian random process. Local topography with large gusts and long lulls potentially create Non-Gaussian process

2.2. Multivariate spectral representation

(1) For simulation of the turbulent wind field at external-subjected-loading nodes on bridge deck, some attentions should be paid for simplification:
   a) Only horizontally and vertically fluctuating wind fields are practically required at deck nodes. Moreover, these wind fields are independent each other.
   b) Wind loading at tower nodes and cable nodes is negligible.
   c) Wind field is assumed as homogeneous along bridge deck.
   d) Nodal elevation is equal along bridge deck.
   e) Loading-subjected nodes space at every equal interval along bridge deck.

(2) Simulation of wind fluctuations for bridges can be considered as that of multi-dimensional stationary random processes in which the horizontal or vertical wind fluctuations are simulated at deck nodes. Apart from the simulation of single-
dimensional random processes, the simulation of multi-dimensional ones requires taking the correlation or inter-influence between processes. In the other word, cross-spectrum of wind fluctuation between nodes must be taken into account.

(3) Let consider a set of m stationary Gaussian random processes \( u_i^0(t), i=1,2,\ldots,m \), with zero mean, having the targeted one-sided cross-spectral density matrix \( S(\omega) \), where superscript 0 denoted the targeted function. The simulation of multi-dimensional processes can be expressed by following equation [Shizuoka(1990)]:

\[
u_i(t) = \sum_{l=1}^{m} \sum_{k=1}^{N} |H_{il}(\omega)| \sqrt{2\Delta \omega} \cos[\omega_k t + \theta_{il}(\omega_k) + \phi_{lk}] \]

Where:

\( H_{il}(\omega) \): Component with entry index il of matrix \( H(\omega) \) which can be determined by either the Cholesky decomposition or modal factorization of the cross-spectral density matrix \( S(\omega) \) through the following formula:

\[ S(\omega) = H(\omega)\overline{H^T}(\omega) \]

\( \overline{H^T}(\omega) \): Transposed matrix of complex conjugate

\( \theta_{il}(\omega) \): Complex angle of \( H_{il}(\omega) \)

\[ \theta_{il}(\omega) = \tan^{-1} \frac{\text{Im}\{H_{il}(\omega)\}}{\text{Re}\{H_{il}(\omega)\}} \]
Assumed that wind turbulence processes act longitudinally on m deck nodes of bridges (Fig.9), the cross-spectral density matrix $S(\omega)$ of m-dimensional process is given by such a form:

$$S(\omega) = \begin{bmatrix}
S_{11}(\omega) & S_{12}(\omega) & \ldots & S_{1m}(\omega) \\
S_{21}(\omega) & S_{22}(\omega) & \ldots & S_{2m}(\omega) \\
\vdots & \vdots & \ddots & \vdots \\
S_{m1}(\omega) & S_{m2}(\omega) & \ldots & S_{mm}(\omega)
\end{bmatrix}$$

Because of $S_{ij}(\omega) = S_{ji}(\omega)$, thus

$$S(\omega) = \begin{bmatrix}
S_{11}(\omega) & \ldots & S_{Sym.} \\
S_{21}(\omega) & S_{22}(\omega) & \ldots \\
\vdots & \vdots & \ddots \\
S_{m1}(\omega) & S_{m2}(\omega) & \ldots & S_{mm}(\omega)
\end{bmatrix}$$

The cross-spectral density function $S_{ij}(\omega)$ between nodes i and j is defined as follow:

$$S_{ij}(\omega) = \sqrt{S_{ii}(\omega)}\sqrt{S_{jj}(\omega)}\text{Coh}_{ij}(\omega)$$

$S_{ii}(\omega), S_{jj}(\omega)$: Auto-spectral density functions at nodes i and j

$\text{Coh}_{ij}(\omega)$: Coherence function

The coherence function between nodes i and j can be determined by the Davenport’s empirical formula:

$$\text{Coh}_{ij}(\omega) = e^{-nf}$$

$$f = \frac{[C_z(z_j - z_i)^2 + C_y(y_j - y_i)^2]^{1/2}}{0.5[U_i + U_j]}$$

$C_z, C_y$: Exponent decay coefficients are experimentally determined.

Under the assumption that the homogeneous wind field, we have $U_i = U_j = U$
Under the assumption that the equal nodal elevation, we have \( z_j = z_i \)

Thus spanwise coherence function is rewritten:
\[
 f = \frac{C_y |y_1 - y_2|}{U} = \frac{C_y \Delta y}{U}
\]

\[
 Coh_{ij}(\omega) = e^{\frac{nC_i \Delta y}{U}}
\]

Davenport (1968): \( 8 \leq C_y \leq 16 \)

In some literatures: \( C_y = 10 \)

Thus, \( Coh_{ij}(\omega) = e^{\frac{10n \Delta y}{U}} \)

We have: \( S_{ij}(\omega) = \sqrt{S_{ii}(\omega)} \sqrt{S_{jj}(\omega)} \exp\left(- \frac{10n \Delta y}{U} \right) \)

\[
 S_{ij}(\omega) = S_{ii} \exp\left(- \frac{10n \Delta y}{U} \right)
\]

\[
 S_{ij}(\omega) = S_{ii} \exp\left(- \frac{5 \omega \Delta y}{\pi U} \right)
\]

Under the assumption that selected nodes space equally along span, the cross-spectral density function can be rewritten as:
\[
 S_{ij}(\omega) = S_{ii} \exp\left(- \frac{5 \omega \Delta |j - i|}{\pi U} \right) = S_{ii} [\exp(- \frac{5 \omega \Delta}{\pi U})]^{|j-i|} = S_{ii} C^{|j-i|}
\]

The cross-spectral density matrix \( S(\omega) \) can be rewritten as follow:
\[
S(\omega) = S_{11}(\omega) = \begin{bmatrix}
1 & \ldots & \text{Sym.} \\
C & 1 & \ldots \\
C^2 & C & 1 & \ldots \\
\vdots & \vdots & \ddots & \vdots \\
C^{m-1} & C^{m-2} & C^{m-3} & \ldots & 1
\end{bmatrix}
\]

Matrix \(H(\omega)\) can be determined by the Cholesky decomposition:

\[
H(\omega) = \sqrt{S_{11}(\omega)}G(\omega)
\]

\[
H(\omega) = \sqrt{S_{11}(\omega)} = \begin{bmatrix}
1 & \ldots & 0 \\
C & \sqrt{1-C^2} & \ldots \\
C^2 & C\sqrt{1-C^2} & \sqrt{1-C^2} & \ldots \\
\vdots & \vdots & \ddots & \vdots \\
C^{m-1} & C^{m-2}\sqrt{1-C^2} & C^{m-3}\sqrt{1-C^2} & \ldots & \sqrt{1-C^2}
\end{bmatrix}
\]

Thus, the wind fluctuating velocities at \(m\) deck nodes can be simulated by following equation:

\[
u_i(t) = \sqrt{2} \Delta \omega \sum_{l=1}^{m} \sum_{k=1}^{N} \sqrt{S_{11}(\omega_k)}G(\omega_k) \cos(\omega_k t + \phi_{lk})
\]

### 2.3. Multivariate time series representation

ARMA\((p,q)\) model is combined by the autoregressive model AR\((p)\) and moving average model MA\((q)\) to represent a time series \(Z(t)\) as follows:

\[
Z_t - \phi_1 Z_{t-1} - \phi_2 Z_{t-2} - \ldots - \phi_p Z_{t-p} = X_t + \theta_1 X_{t-1} + \theta_2 X_{t-2} + \ldots + \theta_q X_{t-q}
\]

AR\((p)\) model: \(Z_t - \phi_1 Z_{t-1} - \phi_2 Z_{t-2} - \ldots - \phi_p Z_{t-p} = X_t\)

MA\((q)\) model: \(Z_t = X_t + \theta_1 X_{t-1} + \theta_2 X_{t-2} + \ldots + \theta_q X_{t-q}\)
Where:
Zt stationary process Z(t) at t,
Xt white noise process X(t) at t, is Gaussian distributed process with zero mean ($\mu = 0$) and unit variance ($\sigma^2 = 1$) [distributed between -1 and 1]

$Z_t, Z_{t-1}, Z_{t-2}, \ldots, Z_{t-p}$: Ensemble of Z(t) at different time lags
$X_t, X_{t-1}, X_{t-2}, \ldots, X_{t-q}$: Ensemble of X(t) at different time lags
$\phi_1, \phi_2, \ldots, \phi_p$: AR coefficients or parameters
$\theta_1, \theta_2, \ldots, \theta_q$: MA coefficients or parameters
p order of AR model; q order of MA model

Simulated time series Zt by ARMA(p,q) model

$$Z_t = \sum_{i=1}^{p} \phi_i Z_{t-i} + X_t + \sum_{i=1}^{q} \theta_i X_{t-i}$$

If simulated time series are expressed under multi-variate process vector, then AR(p) model coefficients $\phi_i$ and MA(q) model coefficients $\theta_i$ become coefficient matrices $A_i, B_i$

$$Z_t = \sum_{i=1}^{p} A_i Z_{t-i} + \sum_{i=1}^{q} B_i X_{t-i}$$
3. TIME DOMAIN BUFFETING FORCES

3.1. Introduction

(1) Unsteady buffeting forces in time domain can be modelled through some following ways:
   a) Quasi-steady buffeting forces
   b) Corrected quasi-steady forces (by aerodynamic admittance)
   c) Indicial function

(2) Aerodynamic admittance as transfer function between: i) Steady-state (quasi-steady) forces and unsteady (total) forces, ii) Wind fluctuations and unsteady forces and iii) PSD of fluctuations and PSD of unsteady forces

(3) Scanlan (1974, 2000, 2001) proved inter-relations between i) Flutter derivatives and aerodynamic admittance, ii) indicial function and aerodynamic admittance

(4) Indicial function and convolution integral (time history form): Description of indicial function by convolution integral and rational function approximation.

\[ L_B = \frac{1}{2} \rho U^2 B \left[ 2C_{L0} \frac{u}{U} + C'_L \frac{W}{U} \right] \]
\[ D_B = \frac{1}{2} \rho U^2 B \left[ C_{D0} \frac{2u}{U} + C'_D \frac{w}{U} \right] \]

\[ M_B = \frac{1}{2} \rho U^2 B^2 \left[ C_{M0} \frac{2u}{U} + C'_M \frac{w}{U} \right] \]

3.2. Corrected quasi-steady buffeting forces

Aerodynamic admittance functions will be added as correction function between quasi-steady forces and unsteady forces:

\[ L_B = \frac{1}{2} \rho U^2 B \left[ 2C_{L0} \chi_{Lu} \frac{u}{U} + C'_L \chi_{Lw} \frac{w}{U} \right] \]

\[ D_B = \frac{1}{2} \rho U^2 B \left[ C_{D0} \chi_{Du} \frac{2u}{U} + C'_D \chi_{Dw} \frac{w}{U} \right] \]

\[ M_B = \frac{1}{2} \rho U^2 B^2 \left[ C_{M0} \chi_{Mu} \frac{2u}{U} + C'_M \chi_{Mw} \frac{w}{U} \right] \]

Theoretical aerodynamic admittance can be defined as Sears’s function in which Sear’s function \([Fung (1955)]\) expressed by Bessel functions of first kind

\[ \chi(k) = C(k) [J_0(k) - iJ_1(k)] + iJ_1(k) \]

Where: \(C(k)\) Theodorsen’s complex circulation function (complex form)

Horlock’s function (1968): Extension for contribution of horizontal fluctuation on aerodynamic admittance

\[ \chi(k) = \frac{C(k)}{2} [J_0(k) - iJ_1(k)] + iJ_1(k) + \frac{J_0(k)}{2} \]
\[ C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)} \]

\( H_1^{(2)}(k) \) : Hankel function of 2 rank and 1 order;

\( H_0^{(2)}(k) \) : Hankel function of 2 rank and 0 order
3.3. Indicial function

(1) Wagner’s problem (1925):
Expression of aerodynamic forces under form of indicial function (thanks to the change of staedy state from zero value to incremental angle attack $\alpha_0$):

$$L(s) = \frac{1}{2} \rho U^2 BC_L \alpha_0 \varphi(s),$$

$s$: time dimensionless variable ($s = \frac{tU}{B}$)

$\varphi(s)$: Indicial function (Wagner’s function)

By Wagner (1925) for airfoil (or flat plate): $\varphi(0) = 0.5; \varphi(\infty) = 1$

Time history form of lift associated with arbitrary motion $\alpha(s)$ can be expressed thanks to integral of convolution

$$L(s) = \frac{1}{2} \rho U^2 BC_L \int_{-\infty}^{s} \varphi(s - \sigma) \alpha'(\sigma) d\sigma$$

Partial integration:

$$\int_{-\infty}^{s} \varphi(s - \sigma) \alpha'(\sigma) d\sigma = \varphi(s - \sigma) \alpha(\sigma) \bigg|_{-\infty}^{s} - \int_{-\infty}^{s} \alpha(\sigma) d\varphi(s - \sigma)$$
\[ L(s) = \frac{1}{2} \rho U^2 BC_L [\varphi(0)\alpha(s) - \int_0^\infty \varphi'(\sigma)\alpha(s-\sigma)d\sigma] \]

Expression of practical indicial function by rational function

Any monotonic function, upper limit at 1, can be expressed by evolution from:

\[ \Phi(s) = 1 - ae^{-bs} \quad (s \to \infty) \]

\[ \Phi(s) = 1 - ae^{-bs} - ce^{-ds} \quad \text{(most applicable in aeronautical field)} \]

\[ \Phi'(s) = abe^{-bs} + cde^{-ds} \]

(a, b, c, b : Chosen constants)

Jones (1940): \[ \Phi(s) = 1 - 0.165e^{-0.0455s} - 0.335e^{-0.3s} \]

Garrick (1938): \[ \Phi(s) = \frac{s + 2}{s + 4} \quad \text{(Fractional form)} \]
(2) Theodorsen’s problem (1934): For airfoil flutter under sinusoidal gust

\[ L = -\pi \rho UBC(k) \dot{h} \]

\[ L(s) = \frac{1}{2} \rho U^2 BC_L \chi_T(k) \alpha ; \text{in which: } C''_L = -2\pi ; \alpha = \dot{h}/U \]

\[ \chi_T(k) = C(k) = F(k) + iG(k) : \text{Complex aerodynamic admittance} \]

Transform into PSD:

\[ S_L(k) = \left[ \frac{1}{2} \rho U^2 BC_L \right]^2 |\chi_T(k)|^2 S_\alpha(k) \]

\[ |\chi_T(k)|^2 = |C(k)|^2 = F^2(k) + G^2(k) \]

(3) Kussner’s problem (1936): For airfoil in uniform gust

\[ L(s) = \frac{1}{2} \rho UBC_L \psi(s) \chi_T(k) w_0 \]

\( w_0 \): Uniform vertical fluctuating velocity

\( \psi(s) \): Kussner’s function; \( \psi(0) = 0; \psi(\infty) = 1 \)

\[ \psi(s) = 1 - 0.5e^{-0.13s} - 0.5e^{-1.0s} \]

\[ |\psi(s)|^2 : \text{Kussner aerodynamic admittance} \]

\[ |\chi_T(k)|^2 = |C(k)|^2 = F^2(k) + G^2(k) \]
(4) Sear’s problem (1941): For airfoil in sinusoidal gust

\[ L(s) = \frac{1}{2} \rho UBC' \chi_s(k)w_0 e^{iks} \]

\[ \chi_s(k) = \psi(0) + \psi'(k) = F_s(k) + iG_s(k) \]

\[ |\chi_s(k)|^2 = |C(k)|^2 = F_s^2(k) + G_s^2(k) \]

\[ |\chi_s(k)|^2 = \frac{1}{1 + 5k} \quad \text{[by Scanlan(1993)]} \]

(5) Scanlan’s problem (1974): Developed indicial function for application of bluff body

\[ \Phi(s) = 1 + 2.5e^{-0.8s} - 3.0e^{-s} \]
(6) Relation between flutter derivatives and indicial function

Time history of forces can be expressed under form of convolution integral and indicial function

\[
L(s) = \frac{1}{2} \rho U^2 BC_L [\Phi(0)\alpha(s) - \int_0^\infty \Phi'(\sigma)\alpha(s - \sigma) d\sigma]
\]

\[
\Phi'(\sigma) = \frac{d\Phi(\sigma)}{d\sigma}
\]

\[
\alpha(s) = \alpha_0 e^{iks} : \text{Form of complex sinusoidal moving}
\]

\[
L(s) = \frac{1}{2} \rho U^2 BC_L [\Phi(0)\alpha_0 e^{iks} - \int_0^\infty \Phi'(\sigma)\alpha_0 e^{ik(s - \sigma)} d\sigma]
\]
\[ L(s) = \frac{1}{2} \rho U^2 BC_L [\Phi(0) - \int_0^\infty \Phi'(\sigma)e^{-ik\sigma} \, d\sigma] \alpha_0 e^{iks} \]

\[ L(s) = \frac{1}{2} \rho U^2 BC_L [\Phi(0) - \overline{\Phi}'] \alpha_0 e^{iks} \]

Where: \[ \overline{\Phi}'(k) = \int_0^\infty \Phi'(\sigma)e^{-ik\sigma} \, d\sigma \]

Fourier Transform of first derivative of indicial function

Self-excited forces:

\[ L(k) = \frac{1}{2} \rho U^2 B[KH_1^* \frac{\dot{h}}{U} + KH_2^* \frac{B\dot{\alpha}}{U} + K^2 H_3^* \alpha + K^2 H_4^* \frac{h}{B}] \]

\[ \alpha(s) = \alpha_0 e^{iks} \]

\[ h(s) = h_0 e^{iks} \]

\[ \alpha(s) \approx \frac{\dot{h}(s)}{U} \] (Approximation with small rotation)

\[ L(s) = \frac{1}{2} \rho U^2 BK[H_1^* - iH_4^*] \alpha_0 e^{iks} \]

Thus,

\[ L(s) = \frac{1}{2} \rho U^2 BC_L [\Phi_L(0) - \overline{\Phi}'_L] \alpha_0 e^{iks} \]
\[ C_L' [\Phi_{Lh} (0) - \overline{\Phi'}_{Lh} ] = K [H_1^* - iH_4^*] \]

Similarly,
\[ C_L' [\Phi_{Lx} (0) - \overline{\Phi'}_{Lx} ] = K^2 [H_3^* + iH_2^*] \]
\[ C_D' [\Phi_{Dp} (0) - \overline{\Phi'}_{Dp} ] = K [P_1^* - iP_4^*] \]
\[ C_D' [\Phi_{Dx} (0) - \overline{\Phi'}_{Dx} ] = K^2 [P_3^* + iH_2^*] \]
\[ C_M' [\Phi_{Mh} (0) - \overline{\Phi'}_{Mh} ] = K [A_1^* - iA_4^*] \]
\[ C_D' [\Phi_{Mx} (0) - \overline{\Phi'}_{Mx} ] = K^2 [A_3^* + iA_2^*] \]
4. TIME DOMAIN SELF-EXCITED FORCES

4.1. Models of self-excited forces

(1) Model 1: Self-excited forces per unit deck length of bluff sections can be expressed in frequency domain thanks to experimentally-determined flutter derivatives (Scanlan, 1971).

\[ L_{se} = \frac{1}{2} \rho B U^2 \left\{ K H_1^*(K) \frac{\dot{h}}{U} + K H_2^*(K) \frac{B \dot{\alpha}}{U} + K^2 H_3^*(K) \alpha + K^2 H_4^*(K) \frac{h}{B} \right\} \]

\[ D_{se} = \frac{1}{2} \rho B U^2 \left\{ K P_1^*(K) \frac{\dot{p}}{U} + K P_2^*(K) \frac{B \dot{\alpha}}{U} + K^2 P_3^*(K) \alpha + K^2 P_4^*(K) \frac{p}{B} \right\} \]

\[ M_{se} = \frac{1}{2} \rho B^2 U^2 \left\{ K A_1^*(K) \frac{\dot{h}}{U} + K A_2^*(K) \frac{B \dot{\alpha}}{U} + K^2 A_3^*(K) \alpha + K^2 A_4^*(K) \frac{h}{B} \right\} \]

\[ H_1^*, H_4^*, A_2^*, A_3^*, P_1^*, P_4^* : \text{uncoupled derivatives} \]

\[ H_2^*, H_3^*, A_1^*, A_4^*, P_2^*, P_3^* : \text{coupled derivatives} \]

(2) Model 2: Self-excited forces can be expressed in the time domain thanks to indicial functions (Scanlan, 1974)

\[ L(s) = \frac{1}{2} \rho U^2 B [C_L^* + C_L^\prime] \Phi_L(s) \alpha(s) \]

\[ D(s) = \frac{1}{2} \rho U^2 B [C_D^* + C_D^\prime] \Phi_D(s) \alpha(s) \]

\[ M(s) = \frac{1}{2} \rho U^2 B [C_M^* + C_M^\prime] \Phi_M(s) \alpha(s) \]
Time history form of lift associated with arbitrary motion $\alpha(s)$ can be expressed thanks to convolution integral

$$L(s) = \frac{1}{2} \rho U^2 B[C'_L + C_L] \int_{-\infty}^{s} \Phi_L(s - \sigma) \alpha'(\sigma) d\sigma$$

$$L(s) = \frac{1}{2} \rho U^2 B[C'_L + C_L][\Phi_L(0)\alpha(s) - \int_{0}^{\infty} \Phi'_L(\sigma)\alpha(s - \sigma) d\sigma]$$

$$\Phi'_L(\sigma) = \frac{d\Phi_L(\sigma)}{d\sigma}$$

(3) Model 3: Time history of self-excited forces can be expressed in time domain thanks to convolution integrals of impulse response functions basing on impulse displacements (Lin et al., 1988).

$$L_{se}(t) = L_h + L_{\alpha} = \frac{1}{2} \rho U^2 \int_{-\infty}^{t} f_{Lh}(t - \tau)h(\tau)d\tau + \int_{-\infty}^{t} f_{L\alpha}(t - \tau)\alpha(\tau)d\tau$$

$$D_{se}(t) = D_p + D_{\alpha} = \frac{1}{2} \rho U^2 \int_{-\infty}^{t} f_{Dh}(t - \tau)p(\tau)d\tau + \int_{-\infty}^{t} f_{D\alpha}(t - \tau)\alpha(\tau)d\tau$$

$$M_{se}(t) = M_h + M_{\alpha} = \frac{1}{2} \rho U^2 \int_{-\infty}^{t} f_{Mh}(t - \tau)h(\tau)d\tau + \int_{-\infty}^{t} f_{M\alpha}(t - \tau)\alpha(\tau)d\tau$$

$f_{Lh}, f_{L\alpha}$: Impulse response function of lift forces associated with impulse displacement components: vertical h and rotational $\alpha$

$\tau$: Time delay
4.2. Impulse response function of self-excited forces

Fourier transform of frequency domain self-excited forces

\[ M_{se} = \frac{1}{2} \rho B^2 U^2 \left\{ KA_1^*(K) \frac{\dot{h}}{U} + KA_2^*(K) \frac{B \dot{\alpha}}{U} + K^2 A_3^*(K) \alpha + K^2 H_4^*(K) \frac{h}{B} \right\} \]

\[ M_{se} = \frac{1}{2} \rho B^3 [\omega A_1^* \dot{h} + \omega^2 A_4^* h] + \frac{1}{2} \rho B^4 [\omega A_2^* \dot{\alpha} + \omega^2 A_3^* \alpha] \]

Harmonic oscillation, \( \dot{h} = i \omega \; \text{h}; \dot{\alpha} = i \omega \alpha \)

\[ M_{se} = \frac{1}{2} \rho B^3 \omega^2 [iA_1^* + A_4^*] h + \frac{1}{2} \rho B^4 \omega^2 [iA_2^* + A_3^*] \alpha \]

\[ F_{M_{se}}(\omega) = \frac{1}{2} \rho B^3 \omega^2 [iA_1^* + A_4^*] \int_{-\infty}^{\infty} h(\tau) \exp(i \omega \tau) d\tau \]

\[ + \frac{1}{2} \rho B^4 \omega^2 [iA_2^* + A_3^*] \int_{-\infty}^{\infty} \alpha(\tau) \exp(i \omega \tau) d\tau \]

\[ F_{M_{se}}(\omega) = \frac{1}{2} \rho B^3 \omega^2 [iA_1^* + A_4^*] F_{h(\tau)} + \frac{1}{2} \rho B^4 \omega^2 [iA_2^* + A_3^*] F_{\alpha(\tau)} \] (1)

Fourier transform of time domain self-excited forces

\[ M_{se}(t) = M_{h} + M_{\alpha} = \frac{1}{2} \rho U^2 \int_{-\infty}^{t} f_{Mh}(t-\tau) h(\tau) d\tau + \int_{-\infty}^{t} f_{M\alpha}(t-\tau) \alpha(\tau) d\tau \]

\[ F_{M_{se}} = F_{M_{h}} + F_{M_{\alpha}} = \frac{1}{2} \rho U^2 \int_{-\infty}^{t} \int_{-\infty}^{t} f_{Mh}(t-\tau) h(\tau) \exp(i \omega \tau) d\tau d\tau \]

\[ + \frac{1}{2} \rho U^2 \int_{-\infty}^{t} \int_{-\infty}^{t} f_{M\alpha}(t-\tau) \alpha(\tau) \exp(i \omega \tau) d\tau d\tau \]
\[ F_{M_{se}} = F_{M_{h}} + F_{M_{a}} = \frac{1}{2} \rho U^2 \int_{-\infty}^{\infty} f_{M_{h}}(\tau) \exp(i\omega \tau) d\tau \int_{-\infty}^{\infty} h(\tau) \exp(i\omega \tau) d\tau \]

\[ \frac{1}{2} \rho U^2 \int_{-\infty}^{\infty} f_{M_{a}}(\tau) \exp(i\omega \tau) d\tau \int_{-\infty}^{t} \alpha(\tau) \exp(i\omega \tau) d\tau \]

\[ F_{M_{se}}(\omega) = F_{M_{h}}(\omega) + F_{M_{a}}(\omega) = \frac{1}{2} \rho U^2 [F_{f_{M_{h}}}(\omega)F_{h}(\omega) + F_{f_{M_{a}}}(\omega)F_{\alpha}(\omega)] \]

(2)

\[ F_{f_{M_{h}}}(\omega), F_{f_{M_{a}}}(\omega) : \text{Role as Transfer functions in DSP} \]

Comparison between (1) and (2),

\[ F_{f_{M_{h}}}(\omega) = \frac{B^3}{U^2} \omega^2 [iA_1^* + A_4^*] = BK^2 [iA_1^* + A_4^*] \]

\[ F_{f_{M_{a}}}(\omega) = B^2 K^2 [iA_2^* + A_3^*] \]

\[ F_{f_{L_{h}}}(\omega) = K^2 [iH_1^* + H_4^*] \]

\[ F_{f_{L_{a}}}(\omega) = BK^2 [iH_2^* + H_3^*] \]

\[ F_{f_{D_{p}}}(\omega) = K^2 [iP_1^* + P_4^*] \]

\[ F_{f_{D_{a}}}(\omega) = BK^2 [iP_2^* + P_3^*] \]

Note:

1) Calculating Impulse response functions basing on flutter derivatives

2) Flutter derivatives are discrete values, however, impulse response functions must be continuous functions (time history expression)
3) Flutter derivatives must be used to approximate to continuous functions thanks to rational function approximation

4.3. Rational function approximation method

According to Roger(1977), Bucher&Lin(1988), Chen&Matsumoto(2000), aerodynamic transfer function can be expressed as rational function approximation of non-dimensionless Laplace variable \( p = \frac{iB\omega}{U} = iK \) (Origin from experimentally sytem identification)

\[
p(s) = A_0 p^0 + A_1 p^1 + A_2 p^2 + \sum_{l=3}^{m} A_l \frac{p}{p + b_l}
\]

\[
p(s) = A_0 + A_1 \left( \frac{i\omega B}{U} \right) + A_2 \left( \frac{i\omega B}{U} \right)^2 + \sum_{l=3}^{m} A_l \frac{(i\omega)}{(i\omega) + b_l \frac{U}{B}}
\]

\[
p(s) = A_0 + A_1 (iK) + A_2 (iK)^2 + \sum_{l=3}^{m} A_l \frac{(iK)}{(iK) + b_l}
\]

\( A_0, A_1, A_2, A_l, b_l \): Arbitrary real frequency-independent contants

\[
\begin{align*}
F_{f_{Lh}} & F_{f_{La}} & K^2[iH_1^* + H_4^*] & BK^2[iH_2^* + H_3^*] \\
F_{f_{DP}} & F_{f_{Da}} & = K^2[iP_1^* + P_4^*] & BK^2[iP_2^* + P_3^*] \\
F_{f_{Mh}} & F_{f_{Ma}} & BK^2[iA_1^* + A_4^*] & B^2K^2[iA_2^* + A_3^*]
\end{align*}
\]
\[ = A_0 + A_1 \left( \frac{i\omega B}{U} \right) + A_2 \left( \frac{i\omega B}{U} \right)^2 + \sum_{l=3}^{m} A_l \frac{i\omega}{i\omega + b_l \frac{U}{B}} \]

Determination of model parameters \( A_0, A_1, A_2, A_l, b_l \) (Real coefficients) by Least Square Method

1) \( F_{f_{lh}} (\omega) = K^2 [iH_1^* + H_4^*] = a_0 + a_1 iK - a_2 K^2 + \sum_{l=3}^{m} a_l \frac{i\omega}{i\omega + b_l \frac{U}{B}} \)

We have:

\[
\frac{i\omega}{i\omega + b_l \frac{U}{B}} = \frac{i\omega(i\omega - b_l \frac{U}{B})}{(i\omega + b_l \frac{U}{B})(i\omega - b_l \frac{U}{B})} = -\omega^2 - i\omega \frac{U}{B} \frac{b_l}{B^2}
\]

\[
1 + i \frac{U}{\omega B} b_l = \frac{4\pi^2 + i2\pi b_l V}{4\pi^2 + b_l^2 V^2}
\]

\[
F_{f_{lh}} (\omega) = K^2 [iH_1^* + H_4^*] = a_0 + a_1 iK - a_2 K^2 + \sum_{l=3}^{m} a_l \frac{4\pi^2 + i2\pi b_l V}{4\pi^2 + b_l^2 V^2}
\]

\[
iH_1^* + H_4^* = \frac{V^2}{4\pi^2} \left[ a_0 + a_1 \frac{i2\pi}{V} - a_2 \frac{4\pi^2}{V^2} + \sum_{l=3}^{m} a_l \frac{4\pi^2 + i2\pi b_l V}{4\pi^2 + b_l^2 V^2} \right]
\]

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\[ a_0 \frac{V^2}{4\pi^2} + ia_1 \frac{V}{2\pi} - a_2 + \sum_{l=3}^{m} a_i \frac{V^2}{4\pi^2 + b_l^2V^2} + i\sum_{l=3}^{m} a_l \frac{b_lV^3}{2\pi(4\pi^2 + b_l^2V^2)} \]

\[ = [a_0 \frac{V^2}{4\pi^2} - a_2 + \sum_{l=3}^{m} a_l \frac{V^2}{4\pi^2 + b_l^2V^2}] + i[a_1 \frac{V}{2\pi} + \sum_{l=3}^{m} a_l \frac{b_lV^3}{2\pi(4\pi^2 + b_l^2V^2)}] \]

Thus,

\[ H_1^* = a_1 \frac{V}{2\pi} + \sum_{l=3}^{m} a_l \frac{b_lV^3}{2\pi(4\pi^2 + b_l^2V^2)} \]

\[ H_4^* = a_0 \frac{V^2}{4\pi^2} - a_2 + \sum_{l=3}^{m} a_l \frac{V^2}{4\pi^2 + b_l^2V^2} \]

Similarly,

2)

\[ F_{f_{la}} (\omega) = K^2[iH_2^* + H_3^*]B = a_0 + a_1iK - a_2K^2 + \sum_{l=3}^{m} a_l \frac{i\omega}{i\omega + b_l \frac{U}{B}} \]

\[ H_2^* = a_1 \frac{V}{2\pi B} + \sum_{l=3}^{m} a_l \frac{b_lV^3}{B(4\pi^2 + b_l^2V^2)} \]

\[ H_4^* = a_0 \frac{V^2}{4\pi^2 B} - \frac{a_2}{B} + \sum_{l=3}^{m} a_l \frac{V^2}{B(4\pi^2 + b_l^2V^2)} \]

3)
\[ F_{f_{Mh}} (\omega) = K^2 [iA_1^* + A_4^*]B = a_0 + a_1 iK - a_2 K^2 + \sum_{l=3}^m a_l \frac{i \omega}{i \omega + b_l} \frac{U}{B} \]

\[ A_1^* = a_1 \frac{V}{2\pi B} + \sum_{l=3}^m a_l \frac{b_l V^3}{B(4\pi^2 + b_l^2 V^2)} \]

\[ A_4^* = a_0 \frac{V^2}{4\pi^2 B} - a_2 \frac{V^2}{B} + \sum_{l=3}^m a_l \frac{V^2}{B(4\pi^2 + b_l^2 V^2)} \]

\[ F_{f_{Ma}} (\omega) = K^2 [iA_2^* + A_3^*]B^2 = a_0 + a_1 iK - a_2 K^2 + \sum_{l=3}^m a_l \frac{i \omega}{i \omega + b_l} \frac{U}{B} \]

\[ A_2^* = a_1 \frac{V}{2\pi B^2} + \sum_{l=3}^m a_l \frac{b_l V^3}{B^2(4\pi^2 + b_l^2 V^2)} \]

\[ A_3^* = a_0 \frac{V^2}{4\pi^2 B^2} - a_2 \frac{V^2}{B^2} + \sum_{l=3}^m a_l \frac{V^2}{B^2(4\pi^2 + b_l^2 V^2)} \]

Note: Here transfer functions contain \( B, B^2 \)
4.4. Impulse response function (IRF)

Transfer functions $F_{f_{Lh}}(\omega), F_{f_{La}}(\omega), F_{f_{Mh}}(\omega), F_{f_{Ma}}(\omega)$ (Fourier transform of impulse response function) can be expressed under forms of rational functions

$$F_{f_{Lh}}(\omega) = a_0 + a_1iK - a_2K^2 + \sum_{l=3}^{m} a_l \frac{i\omega}{i\omega + b_l \frac{U}{B}}$$

Impulse response function can be determined by Inverse Fourier transform of transfer functions as follows:

$$f_{Lh}(t) = \int_{-\infty}^{\infty} F_{f_{Lh}}(\omega) \exp(-i\omega\tau) d\omega$$

$$f_{Lh}(t) = \int_{-\infty}^{\infty} \left[a_0 + a_1iK - a_2K^2 + \sum_{l=3}^{m} a_l \frac{i\omega}{i\omega + b_l \frac{U}{B}} \right] \exp(-i\omega\tau) d\omega$$
5. RATIONAL FUNCTION APPROXIMATION
OF SELF-EXCITED FORCES: EXAMPLE

5.1. Rational function approximation

\[
F_{f_{Lh}}(\omega) = F_{f_{La}}(\omega) = K^2[iH_1^* + H_4^*], \quad BK^2[iH_2^* + H_3^*]
\]
\[
F_{f_{Dp}}(\omega) = F_{f_{Da}}(\omega) = K^2[iP_1^* + P_4^*], \quad BK^2[iP_2^* + P_3^*]
\]
\[
F_{f_{Mh}}(\omega) = F_{f_{Ma}}(\omega) = BK^2[iA_1^* + A_4^*], \quad B^2K^2[iA_2^* + A_3^*]
\]
\[
= A_0 + A_1 \left( \frac{i\omega B}{U} \right) + A_2 \left( \frac{i\omega B}{U} \right)^2 + \sum_{l=3}^{m} A_l \frac{i\omega}{i\omega + b_l \frac{U}{B}}
\]
\[
\sum_{l=3}^{m} A_l \frac{i\omega}{i\omega + b_l \frac{U}{B}} = \sum_{l=1}^{m} A_l \frac{4\pi^2 + i2\pi b_l V}{4\pi^2 + b_l^2 V^2}
\]
\[
A_0 + A_1 \left( \frac{i\omega B}{U} \right) + A_2 \left( \frac{i\omega B}{U} \right)^2 + \sum_{l=3}^{m} A_l \frac{i\omega}{i\omega + b_l \frac{U}{B}}
\]
\[
= A_0 + A_1 (iK) + A_2 (iK)^2 + \sum_{l=3}^{m} A_l \frac{4\pi^2}{4\pi^2 + b_l^2 V^2} + i \sum_{l=3}^{m} \frac{2\pi b_l V}{4\pi^2 + b_l^2 V^2}
\]
\[
= [A_0 - A_2 K^2 + \sum_{l=3}^{m} A_l \frac{4\pi^2}{4\pi^2 + b_l^2 V^2}] + i[A_1 K + \sum_{l=3}^{m} \frac{2\pi b_l V}{4\pi^2 + b_l^2 V^2}]
\]
\[ F_{f_{Lh}} (\omega) = K^2 [iH_1^* + H_4^*] \]

\[ H_1^* = a_1 \frac{V}{2\pi} + \sum_{l=3}^m a_l \frac{b_l V^3}{2\pi (4\pi^2 + b_l^2 V^2)} \]

\[ H_4^* = a_0 \frac{V^2}{4\pi^2} - a_2 + \sum_{l=3}^m a_l \frac{V^2}{4\pi^2 + b_l^2 V^2} \]

\[ F_{f_{La}} (\omega) = BK^2 [iH_2^* + H_3^*] \]

\[ H_2^* = a_1 \frac{V}{2\pi B} + \sum_{l=3}^m a_l \frac{b_l V^3}{B(4\pi^2 + b_l^2 V^2)} \]

\[ H_4^* = a_0 \frac{V^2}{4\pi^2 B} - a_2 + \sum_{l=3}^m a_l \frac{V^2}{B(4\pi^2 + b_l^2 V^2)} \]

\[ F_{f_{Mh}} (\omega) = BK^2 [iA_1^* + A_4^*] \]

\[ A_1^* = a_1 \frac{V}{2\pi B} + \sum_{l=3}^m a_l \frac{b_l V^3}{B(4\pi^2 + b_l^2 V^2)} \]

\[ A_4^* = a_0 \frac{V^2}{4\pi^2 B} - a_2 + \sum_{l=3}^m a_l \frac{V^2}{B(4\pi^2 + b_l^2 V^2)} \]

\[ F_{f_{Ma}} (\omega) = B^2 K^2 [iA_2^* + A_3^*] \]
\[ A_2^* = a_1 \frac{V}{2\pi B^2} + \sum_{l=3}^{m} a_l \frac{b_l V^3}{B^2 (4\pi^2 + b_l^2 V^2)} \]

\[ A_3^* = a_0 \frac{V^2}{4\pi^2 B^2} - \frac{a_2}{B^2} + \sum_{l=3}^{m} a_l \frac{V^2}{B^2 (4\pi^2 + b_l^2 V^2)} \]

\[ m=3 \text{ taken in example} \]

**5.2. Numerical example**

Approximation of rational functions based on flutter derivatives of flat plate

Flutter derivatives of flat plate determined by quasi-steady formulation [Matsumoto (2000)]

\[ H_1^* = -\frac{\pi F}{k}; H_2^* = -\frac{\pi}{2} \left( \frac{1}{2k} + \frac{G}{k^2} + \frac{F}{2k} \right); H_3^* = -\frac{\pi}{2} (\frac{F}{k^2} - \frac{G}{2k}); H_4^* = \pi \left( \frac{1}{2} + \frac{G}{k} \right) \]

\[ A_1^* = \frac{\pi F}{4k}; A_2^* = \frac{\pi}{2} \left( \frac{1}{8k} + \frac{G}{4k^2} + \frac{F}{8k} \right); A_3^* = \frac{\pi}{2} \left( \frac{F}{4k^2} - \frac{G}{8k} \right); A_4^* = -\frac{\pi G}{4k} \]

\[ k = \frac{K}{2} \]
Least square method is applied to determine real constant coefficients: $A_0$, $A_1$, $A_2$, $A_3$

\[ \begin{bmatrix} F_{f_{Lh}}(\omega) & F_{f_{La}}(\omega) \\ F_{f_{Dp}}(\omega) & F_{f_{Da}}(\omega) \\ F_{f_{Mh}}(\omega) & F_{f_{Ma}}(\omega) \end{bmatrix} = \begin{bmatrix} K^2[iH_1^* + H_4^*] & BK^2[iH_2^* + H_3^*] \\ K^2[iP_1^* + P_4^*] & BK^2[iP_2^* + P_3^*] \\ BK^2[iA_1^* + A_4^*] & B^2K^2[iA_2^* + A_3^*] \end{bmatrix} \]

\[
= A_0 + A_1 \left( \frac{i\omega B}{U} \right) + A_2 \left( \frac{i\omega B}{U} \right)^2 + A_3 \frac{i\omega}{i\omega + 0.5 \frac{U}{B}}
\]

\[
A_0 = \begin{bmatrix} -0.10756 \\ 0.53791 \end{bmatrix}; \quad A_1 = \begin{bmatrix} -1.5626 \\ 7.813 \end{bmatrix}; \quad A_2 = \begin{bmatrix} -1.5687 \\ -0.0005311 \end{bmatrix}; \quad A_3 = \begin{bmatrix} -0.28774 \\ 1.4387 \end{bmatrix}
\]

\[
F_{f_{Lh}}(\omega) = -0.10756 - 1.5526(iK) - 1.5687(iK)^2 - 0.28774 \frac{iK}{iK + 0.5}
\]

\[
F_{f_{La}}(\omega) = -22.747 - 25.112(iK) - 0.005388(iK)^2 + 7.60205 \frac{iK}{iK + 0.5}
\]

\[
F_{f_{Mh}}(\omega) = 0.53791 + 7.813(iK) - 0.0005311(iK)^2 + 1.4387 \frac{iK}{iK + 0.5}
\]

\[
F_{f_{Ma}}(\omega) = 113.73 - 31.523(iK) - 0.001347(iK)^2 - 38.007 \frac{iK}{iK + 0.5}
\]
Rational function approximation of flutter derivatives $H_i$ ($i=1-4$)

Fourier transform of impulse response function associated with lift force
Rational function approximation of flutter derivatives $A_i (i=1-4)$

Fourier transform of impulse response function associated with moment